

BIFURCATION RESULTS FOR THE YAMABE PROBLEM ON RIEMANNIAN MANIFOLDS WITH BOUNDARY

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ABSTRACT. We consider the product of a compact Riemannian manifold without boundary and null scalar curvature with a compact Riemannian manifold with boundary, null scalar curvature and constant mean curvature on the boundary. We use bifurcation theory to prove the existence of a infinite number of conformal classes with at least two non-homothetic Riemannian metrics of null scalar curvature and constant mean curvature of the boundary on the product manifold.

1. INTRODUCTION

Let (M, g) be a compact Riemannian manifold without boundary of dimension $m \geq 3$. It is well know that the metrics of constant scalar curvature in the conformal class $[g]$ of g can be characterized variationally as critical points of the Hilbert-Einstein functional on the conformal class $[g]$. The existence of metrics with constant scalar curvature in conformal class $[g]$ was established trough the combined works of Yamabe [18], Trudinger [17], Aubin [3] and Schoen [15].

Once the existence of constant scalar curvature metrics conformal to g has been established, it is natural do ask about its uniqueness. For instance, if the Yamabe constant $Y[M, [g]]$ defined as the minimum of the normalized Hilbert-Einstein functional on $[g]$ is non-positive then there exists only one of these metrics (up to homotheties). Also, if g is an Einstein metric which is not conformal to the round metric on the sphere \mathbb{S}^n , there exists one unique metric of unit volume and constant scalar curvature in $[g]$ by a result of Obata [13]. However, Pollack proved in [14] that if the Yamabe constant $Y[M, [g]]$ is positive then there are conformal classes with an arbitrarily large number of metrics of unit volume and constant scalar curvature sufficiently C^0 - close to $[g]$.

Bifurcation techniques have been successfully used to study multiplicity of solutions for the Yamabe problem, both in the compact and the noncompact case, see for instance [7], [5] and [6]. In particular, Lima et al. in [12] using bifurcation techniques proved a result of multiplicity in an infinite number of conformal classes on product manifolds.

For compact Riemannian manifolds with boundary and dimension $m \geq 3$ similar problems to those mentioned above have been studied by different authors. For instance, Escobar proved in [?] that almost every compact Riemannian manifold with boundary is conformally equivalent to a with null scalar curvature and constant mean curvature on the boundary. Uniqueness results were obtained by Escobar in [11].

Motivated by the results in [12] in this paper using the bifurcation theory we study the multiplicity of metrics with null scalar curvature and constant mean curvature on conformal class of a product metric. More precisely, consider $(M_1, g^{(1)})$ a compact Riemannian manifold, without boundary, null scalar curvature and $(M_2, g^{(2)})$ a compact Riemannian manifold with boundary, null scalar curvature and constant mean curvature. Consider now the product manifold $M = M_1 \times M_2$ and the family of Riemannian metrics $(g_t)_{t>0}$ with null scalar curvature and constant mean curvature on M defined by $g_t = g^{(1)} \oplus tg^{(2)}$. The main result of the paper (Theorem 11) states that if $(M_2, g^{(2)})$ has positive constant curvature mean on the boundary, then there is a sequence (t_n) strictly decreasing tending to 0 such that every element of this sequence is a bifurcation instant of the family $(g_t)_{t>0}$. For all other value of t the family $(g_t)_{t>0}$ is locally rigid. The fact that every t_n is a bifurcation instant gives an entirely new sequence (g_i) of Riemannian metrics with null scalar curvature and constant mean curvature on boundary such that each $g_i \in [g_t]$ for some t close to t_n with g_i non-isometric to g_t . This proves multiplicity for an infinite number of conformal classes.

The paper is organized as follows: in section 2 we recall some basic fact about variational characterization of the Riemannian metrics with null scalar curvature and constant mean curvature on boundary conformally related to a metric g . In section 3 we give an abstract result of bifurcation and we examine the convergence of a bifurcating branch. Finally, in section 4 we show our main result.

2. VARIATIONAL SETTING OF THE YAMABE PROBLEM

Let (M, g) be a compact Riemannian manifold with boundary and $m = \dim M \geq 3$. We will denote by $\mathbb{H}^1(M)$ the space of functions in $\mathbb{L}^2(M)$ with first weak derivatives in $\mathbb{L}^2(M)$. Endowed with the inner product

$$\langle \varphi_1, \varphi_2 \rangle = \int_M \left(g(\nabla \varphi_1, \nabla \varphi_2) + \varphi_1 \varphi_2 \right) v_g, \quad \varphi_1, \varphi_2 \in \mathbb{H}^1(M),$$

$\mathbb{H}^1(M)$ is a Hilbert space. Let $E : \mathbb{H}^1(M) \longrightarrow \mathbb{R}$ be defined by

$$E(\varphi) = \int_M \left(g(\nabla \varphi, \nabla \varphi) + \frac{m-2}{4(m-1)} R_g \varphi^2 \right) v_g + \frac{m-2}{2} \int_{\partial M} H_g \varphi^2 \sigma_g.$$

It is well known in literature that $\varphi \in \mathbb{H}^1(M)$ is a critical point of E under the constraint $\mathcal{C} = \{ \varphi \in \mathbb{H}^1(M) \mid \int_{\partial M} \varphi^{\frac{2(m-1)}{m-2}} \sigma_g = 1 \}$ if and only if the conformal metric $\tilde{g} = \varphi^{\frac{4}{m-2}} g$ has null scalar curvature and constant mean curvature, see for instance [?, Proposition 2,1].

2.1. The manifold of the normalized harmonic functions. To study the bifurcation of metrics with null scalar curvature and constant mean curvature we will need to characterize these as critical points of E on a special submanifold of $\mathbb{H}^1(M)$. For this purpos, remember that a function $\varphi \in \mathbb{H}^1(M)$ is harmonic if

$$\int_M g(\nabla \varphi, \nabla \phi) v_g = 0,$$

for all $\phi \in \mathbb{H}_0^1(M)$, where $\mathbb{H}_0^1(M)$ is the kernel of trace operator, i.e.,

$$\mathbb{H}_0^1(M) = \{ \phi \in \mathbb{H}^1(M) \mid \phi|_{\partial M} = 0 \}.$$

Let $\mathbb{H}_\Delta^1(M)$ denote the subspace of $\mathbb{H}^1(M)$ given by all harmonic functions. We know that $\mathbb{H}^1(M) = \mathbb{H}_0^1(M) \oplus \mathbb{H}_\Delta^1(M)$. Hence, $\mathbb{H}_\Delta^1(M)$ is a closed subspace of Hilbert space $\mathbb{H}^1(M)$, and therefore an embedded submanifold of it. By the Sobolev embedding, there is a continuous inclusion $\mathbb{H}^1(M) \subset \mathbb{L}^p(\partial M)$ with $p = \frac{2(m-1)}{m-2}$, see for instance [10, Th. 2.21, pg. 19]. Let $\mathcal{H}_1(M, g)$ denote the subset of $\mathbb{H}_\Delta^1(M)$ consisting of those functions φ such that $\int_{\partial M} \varphi^{\frac{2(m-1)}{m-2}} \sigma_g = 1$.

Proposition 1.

- (1) $\mathcal{H}_1(M, g)$ is an embedded codimension 1 submanifold of $\mathbb{H}_\Delta^1(M)$. In addition, for $\varphi \in \mathcal{H}_1(M, g)$ the tangent space $T_\varphi \mathcal{H}_1(M, g)$ of $\mathcal{H}_1(M, g)$ in φ is given by

$$(2.1) \quad T_\varphi \mathcal{H}_1(M, g) = \left\{ \phi \in \mathbb{H}_\Delta^1(M) \mid \int_{\partial M} \varphi^{\frac{m}{m-2}} \phi \sigma_g = 0 \right\}.$$

- (2) Suppose that the metric g has null scalar curvature. Then $\varphi \in \mathcal{H}_1(M, g)$ is a critical point of E on $\mathcal{H}_1(M, g)$ if and only if φ is a smooth function, positive and the conformal metric $\tilde{g} = \varphi^{\frac{4}{m-2}} g$ has null scalar curvature and ∂M has constant \tilde{g} -mean curvature.¹

Assume that $\int_{\partial M} \sigma_g = 1$, and $\varphi_0 = \mathbf{1}^2$ is a critical point of E on $\mathcal{H}_1(M, g)$ (i.e., if g has null scalar curvature and constant mean curvature) then:

- (3) The second variation $d^2 E(\varphi_0)$ of E at φ_0 is given by

$$(2.2) \quad (\phi_1, \phi_2) \longrightarrow 2 \left\{ \int_M g(\nabla \phi_1, \nabla \phi_2) v_g - H_g \int_{\partial M} \phi_1 \phi_2 \sigma_g \right\},$$

where $\phi_i \in \mathbb{H}_\Delta^1(M)$ is such that $\int_{\partial M} \phi_i \sigma_g = 0$, $i = 1, 2$.

- (4) There exists a (unbounded) self-adjoint operator $J_g : \mathbb{L}_2(\partial M) \longrightarrow \mathbb{L}_2(\partial M)$ such that

$$(2.3) \quad d^2 E(\varphi_0)(\phi_1, \phi_2) = 2 \int_{\partial M} J_g(\phi_1|_{\partial M}) \phi_2 \sigma_g,$$

where $\phi_i \in \mathbb{H}_\Delta^1(M)$ satisfies $\int_{\partial M} \phi_i \sigma_g = 0$, $i = 1, 2$.

Proof. Consider the function $\mathcal{V}_g : \mathbb{H}_\Delta^1(M) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{V}_g(\varphi) = \int_{\partial M} \varphi^{\frac{2(m-1)}{m-2}} \sigma_g.$$

In order to prove (1), it is sufficient to show that \mathcal{V}_g is a submersion at φ , for all $\varphi \in \mathcal{H}_1(M, g)$. Note that \mathcal{V}_g is smooth and its differential in $\varphi \in \mathcal{H}_1(M, g)$ is given by $d\mathcal{V}_g(\varphi)\phi = \frac{2(m-1)}{m-2} \int_{\partial M} \varphi^{\frac{m}{m-2}} \phi \sigma_g$, for all $\phi \in \mathbb{H}_\Delta^1(M)$. In particular, for $\phi = \varphi$ we get $d\mathcal{V}_g(\varphi)\phi = \frac{2(m-1)}{m-2} > 0$, therefore $d\mathcal{V}_g(\varphi)$ is surjective. Clearly the kernel of $d\mathcal{V}_g(\varphi)$ is complemented in $\mathbb{H}_\Delta^1(M)$, hence \mathcal{V}_g is a submersion. Now for $\varphi \in \mathcal{H}_1(M, g)$, the tangent space $T_\varphi \mathcal{H}_1(M, g)$ of $\mathcal{H}_1(M, g)$ in φ is the kernel of $d\mathcal{V}_g(\varphi)$ which establishes (2.1).

¹With a slight abuse of terminology, we will say that a metric g on a manifold M with boundary has constant mean curvature meaning that the boundary ∂M has constant mean curvature with respect to g .

²here $\mathbf{1}$ denote the constant function 1 on M

If the metric g has null scalar curvature, it is easy to see that $\varphi \in \mathcal{H}_1(M, g)$ is a critical point of E on $\mathcal{H}_1(M, g)$ if and only if it is a critical point of E under the constraint $\mathcal{C} = \{\varphi \in \mathbb{H}^1(M) \mid \int_{\partial M} \varphi^{\frac{2(m-1)}{m-2}} \sigma_g = 1\}$. This proves (2). Assume $\int_{\partial M} \sigma_g = 1$ and let $\varphi_0 = \mathbf{1}$ be a critical point of E on $\mathcal{H}_1(M, g)$. The formula (2.2) is a straightforward computation based on the method of Lagrange multipliers.

Finally, given $\phi_1, \phi_2 \in \mathbb{H}^1(M)$ we have

$$\int_M g(\nabla \phi_1, \nabla \phi_2) v_g = \int_{\partial M} \mathcal{N}_g(\phi_1|_{\partial M}) \phi_2 \sigma_g,$$

where \mathcal{N}_g denotes the Dirichlet-Neumann map, see for instance [2]. Hence from (2.2) we have

$$d^2 E(\varphi_0)(\phi_1, \phi_2) = 2 \int_{\partial M} (\mathcal{N}_g(\phi_1|_{\partial M}) - H_g \phi_1) \phi_2 \sigma_g,$$

which proves (3) with $J_g(\phi_1) = \mathcal{N}_g(\phi_1|_{\partial M}) - H_g \phi_1$. □

We will call $\mathcal{H}_1(M, g)$ the manifold of normalized harmonic functions.

2.2. Jacobi Operator. The unbounded linear operator $J_g : \mathbb{L}^2(\partial M) \longrightarrow \mathbb{L}^2(\partial M)$ defined by

$$J_g(\phi) = \mathcal{N}_g(\phi) - H_g \phi$$

is called the Jacobi operator. From the equation (2.3) follows that the dimension of $\text{Ker } J_g$ and the number (counted with multiplicity) of negative eigenvalues of J_g are the nullity and the Morse index of φ_0 as a critical point of $d^2 E$ in $\mathcal{H}_1(M, g)$. The spectrum of J_g restricted to $\mathbf{L}_0(M)$, where

$$\mathbf{L}_0(M) = \{\phi \in \mathbb{L}^2(\partial M) \mid \phi = \varphi|_{\partial M}, \varphi \in \mathbb{H}_\Delta^1(M)\},$$

is given by

$$-H_g < \rho_1(\mathcal{N}_g) - H_g \leq \rho_2(\mathcal{N}_g) - H_g \leq \rho_3(\mathcal{N}_g) - H_g \leq \dots,$$

where the $\rho_j(\mathcal{N}_g)$ are repeated according to multiplicity. This proves the following

Proposition 2. *Assume that $\varphi_0 \equiv \mathbf{1}$ is a critical point of $E : \mathcal{H}_1(M, g) \longrightarrow \mathbb{R}$. Then:*

(1) *The Morse index $i(g)$ of φ_0 is given by*

$$i(g) = \max\{j \mid \rho_j(\mathcal{N}_g) < H_g\}.$$

(2) *The nullity of φ_0 , denoted with $\nu(g)$, is*

$$\nu(g) = \dim \text{Ker } J_g.$$

Remark 3. *Observe that $-H_g$ is not included in the spectrum of $J_g|_{\mathbf{L}_0(M)}$ since the only constant function on $\mathbf{L}_0(M)$ is the function null. We also recall the well known fact that eigenfunctions of J_g are smooth and form an orthonormal basis of $\mathbf{L}_0(M)$.*

2.3. Manifold of normalized Riemannian metrics on M . For $k \geq 2$, let $\mathbb{S}^k(M)$ denote the space of \mathcal{C}^k -sections of the vector bundle $T^*M \otimes T^*M$ of symmetric $(0,2)$ -tensors of class \mathcal{C}^k on M . $\mathbb{S}^k(M)$ has natural structure of Banach space. The set of all Riemannian metrics of class \mathcal{C}^k on M $\text{Met}^k(M)$ is an open subset of $\mathbb{S}^k(M)$ and hence an embedding submanifold of it. For all $g \in \text{Met}^k(M)$ the tangent space $T_g \text{Met}^k(M)$ of $\text{Met}^k(M)$ in g is given by $T_g \text{Met}^k(M) = \mathbb{S}^k(M)$ (see [8] and the reference given there for more details on the manifold $\text{Met}^k(M)$).

Let $\mathcal{V} : \text{Met}^k(M) \rightarrow \mathbb{R}$ be defined by

$$\mathcal{V}(g) = \int_{\partial M} \sigma_g.$$

It is easy to check that \mathcal{V} is a submersion and therefore

$$\mathcal{M} = \left\{ g \in \text{Met}^k(M) \mid \int_{\partial M} \sigma_g = 1 \right\},$$

it is an embedding submanifold of $\text{Met}^k(M)$. We will call \mathcal{M} the manifold of normalized Riemannian metrics on M .

3. BIFURCATION OF SOLUTIONS AND CONVERGENCE OF BIFURCATION BRANCHES

Let us consider a smooth curve $[a, b] \ni t \rightarrow g_t \in \mathcal{M}$ with $\varphi_0 = \mathbf{1}$ is a critical point of E on $\mathcal{H}_1(M, g_t)$ for all $t \in [a, b]$ (i.e., g_t has null scalar curvature and constant mean curvature for all t).

Definition 1. Let $[a, b] \ni t \rightarrow g_t \in \mathcal{M}$ be a curve as above. An element $t_* \in [a, b]$ said to be a bifurcation instant for the family $(g_t)_{t \in [a, b]}$ if there exists a sequence $(t_n) \subset [a, b]$ and a sequence $(\varphi_n) \subset \mathbb{H}^1(M)$ such that

- (a) φ_n is a critical point of E on $\mathcal{H}_1(M, g_{t_n})$ for all n ;
- (b) $\varphi_n \neq \mathbf{1}$ for all n ;
- (c) $t_n \rightarrow t_*$ as $n \rightarrow +\infty$;
- (d) $\varphi_n \rightarrow \mathbf{1}$ as $n \rightarrow +\infty$ on $\mathbb{H}^1(M)$.

If $t_* \in [a, b]$ is not a bifurcation instant, then we say that the family $(g_t)_{t \in [a, b]}$ is locally rigid at t_* .

Given $g \in \mathcal{M}$ with null scalar curvature and constant mean curvature H_g , we say that g is nondegenerate if either $H_g = 0$ or H_g is not an eigenvalue of \mathcal{N}_g . The following bifurcation criterion is used to guarantee the conclusion of this work.

Theorem 4. Let M be a compact manifold with boundary, with $m = \dim M \geq 3$. Let $[a, b] \ni t \rightarrow g_t \in \mathcal{M}$ be a smooth curve of Riemannian metrics, with zero scalar curvature and constant mean curvature for all $t \in [a, b]$. Given $t \in [a, b]$ let us denote by \mathbf{n}_t the number of eigenvalues (counted with multiplicity) of \mathcal{N}_{g_t} that are less than H_{g_t} . Suppose that:

- (1) The metrics g_a and g_b are nondegenerate,
- (2) $\mathbf{n}_a \neq \mathbf{n}_b$.

Then, there exists a bifurcation instant $t_* \in]a, b[$ for the family $(g_t)_{t \in [a, b]}$.

Proof. The result follows from the non-equivariant bifurcation theorem [12, Theorem A.2] \square

Remark 5. It follows from implicit function theorem that if t_* is a bifurcation instant for the family $(g_t)_{t \in [a, b]}$, then the mean curvature $H_{g_{t_*}}$ of the metric g_{t_*} is a nonzero eigenvalue of $\mathcal{N}_{g_{t_*}}$, see for instance [12, Prop. 3.1]. An instant $t \in [a, b]$ such that the mean curvature H_{g_t} is a nonzero eigenvalue of \mathcal{N}_{g_t} is called degeneracy instant of $(g_t)_{t \in [a, b]}$.

3.1. Convergence of bifurcation branches. Our purpose in this section is to settle the type of convergence of a bifurcating branch of solutions of the Yamabe problem in manifolds with boundary.

Proposition 6. Let $(g_t)_{t \in [a, b]}$ be a family of Riemannian metrics as above. If t_* is a bifurcation instant for the family $(g_t)_{t \in [a, b]}$ then

- (1) $H_n \rightarrow H_{t_*}$ as $n \rightarrow +\infty$, where H_n denotes the mean curvature of the metric $g_n = \varphi_n^{\frac{4}{m-2}} g_{t_n}$;
- (2) If $m \geq 4$ then $\varphi_n \rightarrow 1$ in $\mathbb{W}^{s, p}$ for all integer s and $p = \frac{2(m-2)}{m}$;
- (3) If $m \geq 4$ then $\varphi_n \rightarrow 1$ on $\mathcal{C}^s(\overline{M})$ for all integer s .

Proof. By Proposition 1 for each $n \in \mathbb{N}$, the conformal metric $g_n = \varphi_n^{\frac{4}{m-2}} g_{t_n}$ has null scalar curvature and constant mean curvature H_n . As φ_n satisfies the equations

$$(3.1) \quad \begin{cases} \Delta_{g_{t_n}} \varphi_n = 0 & \text{in } M, \\ \frac{\partial \varphi_n}{\partial \eta^{t_n}} + \frac{m-2}{2} H_{g_{t_n}} \varphi_n = \frac{m-2}{2} H_n \varphi_n^{\frac{m}{m-2}} & \text{on } \partial M, \end{cases}$$

partial integration in (3.1) yields:

$$(3.2) \quad H_n = \frac{2}{m-2} \int_M g_{t_n}(\nabla^{t_n} \varphi_n, \nabla^{t_n} \varphi_n) v_{g_{t_n}} + H_{g_{t_n}} \int_{\partial M} \varphi_n^2 \sigma_{g_{t_n}},$$

where ∇^{t_n} denotes the gradient operator calculated respect to the metric g_{t_n} . Since $g_{t_n} \rightarrow g_{t_*}$ in the \mathcal{C}^k -topology then $H_{g_{t_n}} \rightarrow H_{g_{t_*}}$, $v_{g_{t_n}} = \psi_n v_{g_{t_*}}$ and $\sigma_{g_{t_n}} = \hat{\psi}_n \sigma_{g_{t_*}}$ where $\psi_n \rightarrow 1$ and $\hat{\psi}_n \rightarrow 1$ in the \mathcal{C}^k -topology. Moreover, since $\varphi_n \rightarrow 1$ in $\mathbb{H}^1(M)$ then

$$\int_M \sum_{i,j=1}^m \frac{\partial \varphi_n}{\partial x^i} \frac{\partial \varphi_n}{\partial x^j} v_{g_{t_*}} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Hence

$$\int_M g_{t_n}(\nabla^{t_n} \varphi_n, \nabla^{t_n} \varphi_n) v_{g_{t_n}} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

On the other hand the continuity of the embedding $\mathbb{H}^1(M)$ into $\mathbb{L}^2(\partial M)$ yields

$$\int_{\partial M} \varphi_n^2 \sigma_{g_{t_*}} \rightarrow 1 \quad \text{as } n \rightarrow +\infty.$$

Thus

$$\int_{\partial M} \varphi_n^2 \sigma_{g_{t_n}} \rightarrow 1 \quad \text{as } n \rightarrow +\infty.$$

Therefore of (3.2) we get

$$H_n \rightarrow H_{t_*} \quad \text{as } n \rightarrow +\infty.$$

In order to prove (2) note that $v_n = \varphi_n - \mathbf{1}$ satisfy

$$(3.3) \quad \begin{cases} \Delta_{g_{t_n}} v_n = 0 & \text{in } M, \\ \frac{\partial v_n}{\partial \eta^{t_n}} + \frac{m-2}{2} H_{g_{t_n}} \varphi_n = \frac{m-2}{2} H_n \varphi_n^{\frac{m}{m-2}} & \text{on } \partial M. \end{cases}$$

It follows from standard elliptic estimates (see for instance [1, Th. 15.2, pag. 704]) that

$$(3.4) \quad \|\varphi_n - \mathbf{1}\|_{\mathbb{W}^{s,p}(M)} \leq C_n \left\{ \left\| -\frac{m-2}{2} H_{g_{t_n}} \varphi_n + \frac{m-2}{2} H_n \varphi_n^{\frac{m}{m-2}} \right\|_{\mathbb{W}^{s-1-\frac{1}{p},p}(\partial M)} + \|\varphi_n - \mathbf{1}\|_{\mathbb{L}^p(M)} \right\},$$

where s is positive integer, $1 \leq p = \frac{2(m-2)}{m} < 2$ and the constant C_n depends only on s , the manifold M , the \mathbb{L}^∞ -norm of the coefficients of the elliptic operator $\Delta_{g_{t_n}}$, the ellipticity constant of $\Delta_{g_{t_n}}$, and the moduli of continuity of the coefficients of $\Delta_{g_{t_n}}$. As g_{t_n} tend to g_{t_*} in the \mathcal{C}^k -topology then the coefficients of $\Delta_{g_{t_n}}$ tend uniformly to the coefficients of the operator $\Delta_{g_{t_*}}$. Thus, in (3.4) we can choose a constant $C_n \equiv C$ that does not depend on n . Trace theorem says that the inclusion $\mathbb{W}^{s,p}(M) \hookrightarrow \mathbb{W}^{s-\frac{1}{p},p}(\partial M)$ is continuous hence of (3.4) we have

$$(3.5) \quad \|\varphi_n - \mathbf{1}\|_{\mathbb{W}^{s,p}(M)} \leq \overline{C} \left\{ \left\| -\frac{m-2}{2} H_{g_{t_n}} \varphi_n + \frac{m-2}{2} H_n \varphi_n^{\frac{m}{m-2}} \right\|_{\mathbb{W}^{s-1,p}(M)} + \|\varphi_n - \mathbf{1}\|_{\mathbb{L}^p(M)} \right\},$$

for some positive constant \overline{C} . Since the inclusion $\mathbb{H}^1(M) \hookrightarrow \mathbb{L}^p(M)$ is continuous, then $\|\varphi_n - \mathbf{1}\|_{\mathbb{L}^p(M)} \rightarrow 0$. Furthermore, the inclusion $\mathbb{H}^1(M) \hookrightarrow \mathbb{W}^{1,p}(M)$ is continuous because $1 \leq p < 2$. Hence for $s = 2$ we get that $\varphi_n \rightarrow \mathbf{1}$ in $\mathbb{W}^{s-1,p}$. We claim that $\varphi_n^{\frac{m}{m-2}} \rightarrow \mathbf{1}$ in $\mathbb{W}^{1,p}(M)$. Indeed:

$$\begin{aligned} \int_M (\varphi_n^{\frac{m}{m-2}} - 1)^{\frac{2(m-2)}{m}} v_{g_{t_*}} &\leq \text{Vol}_{g_{t_*}}(M)^{\frac{2}{m}} \left(\int_M (\varphi_n^{\frac{m}{m-2}} - 1)^2 v_{g_{t_*}} \right)^{\frac{m-2}{m}} \\ &= \text{Vol}_{g_{t_*}}(M)^{\frac{2}{m}} \int_M \left[(\varphi_n^{\frac{2m}{m-2}} - 1) - 2(\varphi_n^{\frac{m}{m-2}} - 1) \right] v_{g_{t_*}}. \end{aligned}$$

The continuity of the inclusions $\mathbb{H}^1(M) \hookrightarrow \mathbb{L}^{\frac{2m}{m-2}}(M)$ and $\mathbb{H}^1(M) \hookrightarrow \mathbb{L}^{\frac{m}{m-2}}(M)$ imply that

$$\int_M (\varphi_n^{\frac{m}{m-2}} - 1)^{\frac{2(m-2)}{m}} v_{g_{t_*}} \rightarrow 0.$$

Moreover for each $1 \leq i \leq m$

$$\begin{aligned} \int_M (\partial_i \varphi_n^{\frac{m}{m-2}})^{\frac{2(m-2)}{m}} v_{g_{t_*}} &= \left(\frac{m}{m-2} \right)^{\frac{m-2}{m}} \int_M \varphi_n^{\frac{4}{m}} (\partial_i \varphi_n)^{\frac{2(m-2)}{m}} v_{g_{t_*}} \\ &\leq \left(\frac{m}{m-2} \right)^{\frac{m-2}{m}} \left(\int_M (\partial_i \varphi_n)^2 v_{g_{t_*}} \right)^{\frac{m-2}{m}} \left(\int_M \varphi_n^2 v_{g_{t_*}} \right)^{\frac{2}{m}}. \end{aligned}$$

As the right side of the above inequality tends to zero then

$$\int_M (\partial_i \varphi_n^{\frac{m}{m-2}})^{\frac{2(m-2)}{m}} v_{g_{t_*}} \rightarrow 0,$$

this proves our claim. By (3.5) we have that $\varphi_n \rightarrow \mathbf{1}$ in $\mathbb{W}^{2,p}(M)$. Using induction on s in equation (3.4) we obtain $\varphi_n \rightarrow \mathbf{1}$ in $\mathbb{W}^{s,p}$ for all integer s . This proves (2).

Finally, the proof of statement (3) is a direct consequence of the continuous inclusion $\mathbb{W}^{r+1,p}(M) \hookrightarrow \mathcal{C}^s(\overline{M})$, which holds when $r > s - 1 + \frac{m}{p}$, see for instance [?, Th. 2.30, pag. 50]. \square

Remark 7. *It is important to stress that, for all $t \in (0, \infty)$, one has $\Delta_{tg} = \frac{1}{t}\Delta$, $H_{tg} = \frac{1}{\sqrt{t}}H_g$ and $\eta^{tg} = \frac{1}{\sqrt{t}}\eta^g$. This means that the spectrum of the operator $\mathcal{N}_g - H_g$ is invariant by homothety of the metric. On the other hand, $\sigma_{tg} = t^{m-1/2}\sigma_g$. When needed, we will normalize metrics to have volume 1 on the boundary, without changing the spectral theory of the operator $\mathcal{N}_g - H_g$.*

4. BIFURCATION OF SOLUTIONS FOR THE YAMABE PROBLEM ON PRODUCT MANIFOLDS

Let $(M_1, g^{(1)})$ be a compact Riemannian manifold without boundary and null scalar curvature, and let $(M_2, g^{(2)})$ be a compact Riemannian manifold with boundary, null scalar curvature and constant mean curvature. Consider the product manifold, $M = M_1 \times M_2$, which boundary is given by $\partial M = M_1 \times \partial M_2$. Let m_1 and m_2 be the dimensions of M_1 and M_2 , respectively, and assume that $m = \dim M = m_1 + m_2 \geq 3$. For each $t \in (0, +\infty)$, define $g_t = g^{(1)} \oplus tg^{(2)}$ a metric on the product manifold M . It is easily see that the Riemannian manifold (M, g_t) have null scalar curvature and constant mean curvature

$$H_{g_t} = \frac{m_2 - 1}{(m - 1)\sqrt{t}}H_{g^{(2)}} = \frac{\hat{H}_{g^{(2)}}}{\sqrt{t}}, \quad \text{for all } t > 0,$$

where $\hat{H}_{g^{(2)}} = \frac{m_2 - 1}{(m - 1)}H_{g^{(2)}}$. Let \mathcal{H}_t denote the subspace of $\mathbb{L}^2(\partial M)$ consisting of functions ψ such that $\psi = \varphi|_{\partial M}$, $\varphi \in \mathbb{H}_\Delta^1(M)$ and $\int_{\partial M} \psi \sigma_{g_t} = 0$ and let \mathcal{E}_t denote the subspace of $\mathbb{L}^2(\partial M)$ of those maps ϕ such that $\int_{\partial M} \phi \sigma_{g_t} = 0$. Let $\mathcal{J}_t : \mathcal{H}_t \rightarrow \mathcal{E}_t$ be the Jacobi operator defined by

$$\mathcal{J}_t(\psi) = \mathcal{N}_{g_t}(\psi) - H_{g_t}\psi.$$

Denote by $0 = \rho^{(0)} < \rho^{(1)} < \rho^{(2)} < \dots$ the sequence of all distinct eigenvalues of $\Delta_{g^{(1)}}$, with geometric multiplicity $\mu^{(i)}$, $i \geq 0$, and by $\rho_j^{(i)}(t)$ the j -ésimo eigenvalue of the problem

$$(4.1) \quad \Delta_{g^{(2)}}\varphi + t\rho^{(i)}\varphi = 0 \text{ in } M_2, \quad \frac{\partial \varphi}{\partial \eta^2} = \rho\varphi \text{ on } \partial M_2,$$

with $t > 0$. We will denote by $\mu_j^{(i)}$ the geometric multiplicity of $\rho_j^{(i)}(t)$ and we emphasize that $\rho_j^{(i)}(t)$'s are not necessarily all distinct. The spectrum $\Sigma(\mathcal{J}_t)$ of the Jacobi operator \mathcal{J}_t is given by

$$\Sigma(\mathcal{J}_t) = \left\{ \frac{\rho_j^{(i)}(t) - \hat{H}_{g^{(2)}}}{\sqrt{t}} \mid i + j > 0, \quad i, j \in \mathbb{N}^* \right\},$$

where $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. We are interested in determining all values of t for which $\rho_j^{(i)}(t) = \hat{H}_{g^{(2)}}$, for some $i, j \in \mathbb{N}^*$ with $i + j > 0$. In this case, the Jacobi operator \mathcal{J}_t is degenerate.

Remark 8.

- (a) For $i = 0$ and $j \in \mathbb{N}^*$, $\rho_j^{(i)}(t)$ is an eigenvalue for $\mathcal{N}_{g^{(2)}}$. Hence if $\hat{H}_{g^{(2)}}$ is an eigenvalue for $\mathcal{N}_{g^{(2)}}$ then the Jacobi operator \mathcal{J}_t is degenerate for all $t > 0$.
- (b) Given $\rho^{(i)} > 0$ and $j \in \mathbb{N}^*$, using the variational characterization of $\rho_j^{(i)}(t)$ it is easily seen that

$$\rho_j^{(i)}(t_1) < \rho_j^{(i)}(t_2),$$

for all $0 < t_1 < t_2$.

Proposition 9. Assume that $\hat{H}_{g^{(2)}}$ is not an eigenvalue for $\mathcal{N}_{g^{(2)}}$. For $i \in \mathbb{N}$, let $\rho_0^{(i)}(t)$ denote the first eigenvalue of the problem (4.1). Then,

- (1) The map $(0, \infty) \ni t \longrightarrow \rho_0^{(i)}(t) \in \mathbb{R}$ is continuous;
- (2) $\rho_0^{(i)}(t) \longrightarrow 0$, as $t \longrightarrow 0$;
- (3) $\rho_0^{(i)}(t) \longrightarrow \infty$, as $t \longrightarrow \infty$;
- (4) If $H_{g^{(2)}} > 0$, there exists a unique $t_i > 0$ satisfying $\rho_0^{(i)}(t_i) = \hat{H}_{g^{(2)}}$.

Proof. Since the map $(0, \infty) \ni t \longrightarrow \rho_0^{(i)}(t) \in \mathbb{R}$ is strictly increasing, in order to establish (1) we will show that given $t_0 > 0$ and any two sequences $(t_n^{(1)})_n, (t_n^{(2)})_n$ such that $t_n^{(1)} \searrow t_0$ and $t_n^{(2)} \nearrow t_0$ we have

$$(4.2) \quad \inf_n \rho_0^{(i)}(t_n^{(1)}) = \rho_0^{(i)}(t_0) = \sup_n \rho_0^{(i)}(t_n^{(2)}).$$

It is clear that $\rho_0^{(i)}(t_0) < \rho_0^{(i)}(t_n^{(1)})$ for all $n \in \mathbb{N}$, hence $\rho_0^{(i)}(t_0) \leq \inf_n \rho_0^{(i)}(t_n^{(1)})$.

On the other hand, by the variational characterization of $\rho_0^{(i)}(t_0)$, there exists $\varphi \in \mathcal{C}^\infty(\overline{M_2})$ such that

$$\rho_0^{(i)}(t_0) = E_{t_0}(\varphi) := \int_{M_2} \left(g(\nabla \varphi, \nabla \varphi) + t_0 \rho^{(i)} \varphi^2 \right) v_{g^{(2)}}.$$

Thus of the variational characterization of $\rho_0^{(i)}(t_n^{(1)})$ follows that

$$\rho_0^{(i)}(t_n^{(1)}) \leq E_{t_n^{(1)}}(\varphi) := \int_{M_2} \left(g(\nabla \varphi, \nabla \varphi) + t_n^{(1)} \rho^{(i)} \varphi^2 \right) v_{g^{(2)}}, \quad \text{for all } n \in \mathbb{N}.$$

Therefore

$$\inf_n \rho_0^{(i)}(t_n^{(1)}) \leq \inf_n E_{t_n^{(1)}}(\varphi) = E_{t_0}(\varphi) = \rho_0^{(i)}(t_0),$$

which proves the left equality in (4.2). We will show now the equality of the right hand side in (4.2). Given the sequence $(t_n^{(2)})_n$ such that $t_n^{(2)} \nearrow t_0$, it is evident that $\sup_n \rho_0^{(i)}(t_n^{(2)}) \leq \rho_0^{(i)}(t_0)$. Given $n \in \mathbb{N}$, let $\varphi_n \in \mathcal{C}^\infty(\overline{M_2})$ be such that

$$\rho_0^{(i)}(t_n^{(2)}) = E_{t_n^{(2)}}(\varphi_n), \quad \int_{\partial M_2} \varphi_n^2 \sigma_{g^{(2)}} = 1, \quad \text{and} \quad \int_{\partial M_2} \varphi_n \sigma_{g^{(2)}} = 0,$$

this φ_n exists because of the variational characterization of $\rho_0^{(i)}(t_n^{(2)})$. Note that $\{\varphi_n\}$ is a bounded subset in $\mathbb{H}^1(M_2)$. Since bounded sets are weakly compact in a Hilbert space, there exists a subsequence of $\{\varphi_n\}$, denoted also by $\{\varphi_n\}$, that converges weakly to a function $\psi \in \mathbb{H}^1(M_2)$. Then $\int_{\partial M_2} \psi \sigma_{g^{(2)}} = 0$. Since the

embeddings $\mathbb{H}^1(M_2)$ into $\mathbb{L}^2(\partial M_2)$ and into $\mathbb{L}^2(M_2)$ are compact, then the sequence $\{\varphi_n\}$ satisfies

$$\int_{\partial M_2} \varphi_n^2 \sigma_{g^{(2)}} \longrightarrow \int_{\partial M_2} \psi^2 \sigma_{g^{(2)}} \text{ and } \int_{M_2} \varphi_n^2 \sigma_{g^{(2)}} \longrightarrow \int_{M_2} \psi^2 \sigma_{g^{(2)}}.$$

Hence $\int_{\partial M_2} \psi^2 \sigma_{g^{(2)}} = 1$. From the variational characterization of $\rho_0^{(i)}(t_0)$ we get $\rho_0^{(i)}(t_0) \leq E_{t_0}(\psi)$. Since

$$\int_{M_2} g(\nabla \psi, \nabla \psi) \sigma_{g^{(2)}} \leq \limsup_n \int_{M_2} g(\nabla \varphi_n^2, \nabla \varphi_n^2) \sigma_{g^{(2)}},$$

then

$$E_{t_0}(\psi) \leq \limsup_n E_{t_n^{(2)}}(\varphi_n) = \limsup_n \rho_0^{(i)}(t_n^{(2)}) = \sup_n \rho_0^{(i)}(t_n^{(2)}),$$

the inequality above establishes the equality of the right hand side in (4.2), which completes the proof of (1). Now we will prove (2). To this end note that $0 \leq \rho_0^{(i)}(t)$ for all $t > 0$. Thus $0 \leq \inf_{t>0} \rho_0^{(i)}(t)$. On the other hand, it is known that there exists $C > 0$ such that

$$\int_{M_2} \varphi^2 v_{g^{(2)}} \leq C \left\{ \int_{M_2} g(\nabla \varphi, \nabla \varphi) v_{g^{(2)}} + \int_{\partial M_2} \varphi^2 \sigma_{g^{(2)}} \right\},$$

for all $\varphi \in \mathbb{H}^1(M_2)$. Thus

$$E_t(\varphi) \leq \int_{M_2} g(\nabla \varphi, \nabla \varphi) v_{g^{(2)}} + t \rho^{(i)} C \left\{ \int_{M_2} g(\nabla \varphi, \nabla \varphi) v_{g^{(2)}} + \int_{\partial M_2} \varphi^2 \sigma_{g^{(2)}} \right\}.$$

Hence

$$\frac{E_t(\varphi)}{\int_{\partial M_2} \varphi^2 \sigma_{g^{(2)}}} \leq (1 + t \rho^{(i)} C) \frac{\int_{M_2} g(\nabla \varphi, \nabla \varphi) v_{g^{(2)}}}{\int_{\partial M_2} \varphi^2 \sigma_{g^{(2)}}} + t \rho^{(i)} C,$$

for all $\varphi \in \mathbb{H}^1(M_2) \setminus \{0\}$. Using the variational characterization of $\rho_0^{(i)}(t)$, it follows from the inequality above that

$$0 \leq \rho_0^{(i)}(t) \leq t \rho^{(i)} C, \quad \text{for all } t > 0.$$

Therefore

$$\rho_0^{(i)}(t) \longrightarrow 0 \quad \text{as } t \longrightarrow 0.$$

In order to prove (3) suppose the assertion is false. Let $\rho \in \mathbb{R}$ be such that $\lim_{t \rightarrow \infty} \rho_0^{(i)}(t) = \rho$. Clearly $\rho_0^{(i)}(t) \leq \rho$. Let $\varphi_t \in \mathcal{C}^\infty(\overline{M_2})$ be such that

$$\rho_0^{(i)}(t) = E_t(\varphi_t) \quad \text{and} \quad \int_{\partial M} \varphi_t^2 \sigma_{g^{(2)}} = 1.$$

We claim that $\int_{M_2} \varphi_t^2 v_{g^{(2)}} \longrightarrow 0$ at $t \longrightarrow \infty$. In fact, otherwise

$$\inf_{t>0} \int_{M_2} \varphi_t^2 v_{g^{(2)}} > 0,$$

hence there exists $t > 0$ such that

$$\rho < t \inf_{t>0} \int_{M_2} \varphi_t^2 v_{g^{(2)}} \leq \rho_0^{(i)}(t),$$

this contradicts the fact that $\rho_0^{(i)}(t) \leq \rho$. However $\{\varphi_t\}_{t>0}$ is a bounded set in $\mathbb{H}^1(M_2)$, thus there exists a sequence $(\varphi_{t_n})_n \subset \{\varphi_t\}_{t>0}$ such that

$$\int_{\partial M_2} \varphi_{t_n}^2 v_{g^{(2)}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

which is impossible since $\int_{\partial M_2} \varphi_{t_n}^2 v_{g^{(2)}} = 1$ for all $n \in \mathbb{N}$. Therefore $\lim_{t \rightarrow \infty} \rho_0^{(i)}(t) = \infty$. Finally, (4) is a direct consequence of (1)–(3). \square

Corollary 10. *Let $(M_1, g^{(1)})$ be a compact Riemannian manifold without boundary and scalar curvature null. Let $(M_2, g^{(2)})$ be a compact Riemannian manifold with boundary, scalar curvature null, and constant mean curvature $H_{g^{(2)}}$. We will denote by M the product manifold $M_1 \times M_2$. Consider the family of Riemannian metrics $g_t = g^{(1)} \oplus tg^{(2)}$, $t > 0$, on M and suppose that $m = \dim M \geq 3$. Then:*

- (a) *If $H_{g^{(2)}} \leq 0$, then the Jacobi operator is nondegenerate for all $t \in]0, \infty[$. In particular, the family $(g_t)_{t>0}$ is locally rigid for all $t > 0$.*
- (b) *If $H_{g^{(2)}} > 0$ and \hat{H}_{g_t} is not a Steklov eigenvalue for $\Delta_{g^{(2)}}$, there exists a strictly decreasing sequence tending to zero consisting of degeneracy instants of the family $(g_t)_{t>0}$. For all other values of t , the Jacobi operator \mathcal{J}_t is nonsingular.*

Proof. (a) follows easily from remark 5. For each $i \in \mathbb{N}$, let t_i denote the unique positive real number such that $\rho_0^{(i)}(t_i) = \hat{H}_{g^{(2)}}$. For $0 < i_1 < i_2$ we obtain

$$\hat{H}_{g^{(2)}} = \rho_0^{(i_1)}(t_{i_1}) < \rho_0^{(i_2)}(t_{i_1}) \leq \rho_0^{(i_2)}(t), \quad t \geq t_{i_1}$$

Therefore $t_{i+1} < t_i$ for all $i \in \mathbb{N}$. Hence the sequence $(t_i)_{i \in \mathbb{N}}$ is strictly decreasing. Clearly the Jacobi operator \mathcal{J}_{t_i} is degenerate for all $i \in \mathbb{N}$. Let $\varphi_i \in \mathcal{C}^\infty(\overline{M_2})$ be such that

$$(4.3) \quad \rho_0^{(i)}(t_i) = E_{t_i}(\varphi_i) \quad \text{and} \quad \int_{\partial M_2} \varphi_i^2 \sigma_{g^{(2)}} = 1.$$

As $\rho_0^{(i)}(t_i) = \hat{H}_{g^{(2)}}$ for all $i \in \mathbb{N}$, from first equality in (4.3) follows that

$$\int_{M_2} g(\nabla \varphi_i, \nabla \varphi_i) v_{g^{(2)}} \leq \hat{H}_{g^{(2)}}, \quad \text{for all } i \in \mathbb{N}.$$

Using the inequality above and the second equality in (4.3), we conclude that $\{\varphi_i\}_{i \in \mathbb{N}}$ is bounded in $\mathbb{H}^1(M_2)$. We claim that $\inf_i \int_{M_2} \varphi_i^2 v_{g^{(2)}} > 0$. In fact, otherwise there exists a sequence $(\varphi_{t_{i_n}})_{n \in \mathbb{N}} \subset \{\varphi_i\}$ which converges weakly to 0 in $\mathbb{H}^1(M_2)$. Since the embedding of $\mathbb{H}_1(M_2)$ into $\mathbb{L}_2(\partial M)$ is compact then $\int_{\partial M_2} \varphi_{t_{i_n}}^2 \sigma_{g^{(2)}} \longrightarrow 0$ as $n \longrightarrow \infty$. This contradicts the fact that $\int_{\partial M_2} \varphi_i^2 \sigma_{g^{(2)}} = 1$ for all $i \in \mathbb{N}$. Thus

$$0 < t_i = \frac{\hat{H}_{g^{(2)}} - \int_{M_2} g(\nabla \varphi_i, \nabla \varphi_i) v_{g^{(2)}}}{\rho_i^i \int_{M_2} \varphi_i^2 v_{g^{(2)}}} \leq \frac{\hat{H}_{g^{(2)}}}{\rho_i^i \inf_i \int_{M_2} \varphi_i^2 v_{g^{(2)}}} \longrightarrow 0,$$

as $i \longrightarrow \infty$. Which proves (b). \square

We are ready for our main result.

Theorem 11. *Let $(M_1, g^{(1)})$ be a compact Riemannian manifold without boundary with scalar curvature zero and $(M_2, g^{(2)})$ a compact Riemannian manifold with boundary, having scalar curvature zero and positive constant mean curvature $H_{g^{(2)}}$. Assume that \hat{H}_{g_t} is not a Steklov eigenvalue for $\Delta_{g^{(2)}}$ and $\dim(M_1) + \dim(M_2) \geq 3$. For all $t \in (0, \infty)$, let $g_t = g^{(1)} \oplus tg^{(2)}$ be the metric on the product manifold with boundary, $M = M_1 \times M_2$. Then there exists a sequence tending to zero consisting of bifurcation instants for the family $(g_t)_{t>0}$. For all other values of $t > 0$ the family $(g_t)_{t>0}$ is locally rigid.*

Proof. Corollary 10 establishes the existence of a strictly decreasing sequence (\hat{t}_n) converging to 0 such that the Jacobi operator $\mathcal{J}_{\hat{t}_n}$ is singular for all $n \in \mathbb{N}$. Note that

$$\hat{H}_{g^{(2)}} = \rho_0^{(1)}(\hat{t}_1) < \rho_0^{(i)}(t) \leq \rho_j^{(i)}(t), \quad i, j \in \mathbb{N}, \text{ and } t > \hat{t}_1.$$

Therefore the Jacobi operator is nonsingular on $]\hat{t}_1, +\infty[$. Our next claim is that there are at most a finite number of degeneracy instants for the family $(g_t)_{t>0}$ into $]\hat{t}_{i+1}, \hat{t}_i[$. Indeed, note that

$$\hat{H}_{g^{(2)}} = \rho_0^{(i+1)}(\hat{t}_{i+1}) < \rho_0^{(i+1)}(t) \leq \rho_j^{(r)}(t), \quad r \geq i+1, \quad t > \hat{t}_{i+1}, \quad j \in \mathbb{N}^*.$$

Hence, if $\rho_j^{(r)}(t) = \hat{H}_{g^{(2)}}$, $t \in]\hat{t}_{i+1}, \hat{t}_i[$, then $1 \leq r \leq i$. In addition, for all r there exists $J(r) \in \mathbb{N}$ such that

$$\hat{H}_{g^{(2)}} < \rho_j^{(r)}(\hat{t}_{i+1}) < \rho_j^{(r)}(t), \quad j > J(r), \quad t > \hat{t}_{i+1}.$$

Let $J_0 = \max_{1 \leq r \leq i} \min J(r)$. Thus, if $\rho_j^{(r)}(t) = \hat{H}_{g^{(2)}}$ with $t \in]\hat{t}_{i+1}, \hat{t}_i[$ then $1 \leq r \leq i$

and $1 \leq j \leq J_0$. Finally, for $i_1 < i_2$ we have that $\rho_j^{(i_1)}(t) < \rho_j^{(i_2)}(t)$ for all t and j . Therefore, for each pair (r, j) there exists at most one $t = t(r, j)$ such that $\rho_j^{(r)}(t) = \hat{H}_{g^{(2)}}$, which proves our claim. Hence the set of all degeneracy instants for the family $(g_t)_{t>0}$ consists of a strictly decreasing sequence $(t_i)_{i \in \mathbb{N}}$ converging to 0. For each $i \in \mathbb{N}$, there exists $\epsilon = \epsilon(i) > 0$ such that

$$[t_{i-\epsilon}, t_{i+\epsilon}] \cap (t_n)_{n \in \mathbb{N}} = \{t_i\}.$$

Moreover, for all i and j the map $t \rightarrow \rho_j^{(i)}(t)$ is strictly increasing. Therefore $\mathbf{n}_{t_{i-\epsilon}} \neq \mathbf{n}_{t_{i+\epsilon}}$. Theorem 4 ensures that every degeneracy instant is a bifurcation instant. \square

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